π

Introduction to Π^{Π} Space

This work introduces Π^{Π} Space or $\mathbb{MATE}$ Space, a fractal mathematical framework that emerges naturally when studying the distribution of prime numbers and the zeros of the Riemann zeta function. Through the Mate$(l, a, n)(x)$ function with base-π logarithms, we demonstrate how the inherent fractal structure forces the condition $\text{Re}(s) = \frac{1}{2}$ for non-trivial zeros, thus providing a new perspective on the Riemann Hypothesis.

Abstract

In mathematics, we are accustomed to developing theories and conjectures in domains like $\mathbb{R}^2$ or $\mathbb{R}^3$, which, while providing a sufficient framework to address most mathematically interesting problems of practical importance, are based more on human convenience than on the deep structure of reality. These spaces are easy to visualize, understand, and analyze, but reality shows us time and again that the universe does not conform to Euclidean simplifications. From fractal geometry in nature to the "irregular" distribution of prime numbers, the self-similar and nonlinear complexity of real systems challenges traditional models.

This work presents $\mathbb{MATE}$ **Space** or Π^{Π} **Space**, an emerging mathematical framework that arises naturally when studying the Riemann Hypothesis from a fractal perspective. The initial motivation was to harmonize two fundamental facts: The distribution of prime numbers and the location of non-trivial zeros of the zeta function $\zeta(s)$ on the critical line $\text{Re}(s) = \frac{1}{2}$.

By interpreting the prime distribution as a "wave" in logarithmic space, we discovered that a **base-π Fourier Transform** revealed hidden resonance patterns. However, this approach only partially captured the underlying structure.

Deeper analysis led us to consider **exponential towers of π** and their logarithmic iterations, resulting in a key tool: the Mate$(l, a, n)(x)$ function. This function acts as a *fractal microscope*, capable of decomposing the self-similarity of primes into successive layers. In its limit ($l \to \infty$), when evaluated at π, Mate$(l, \pi^{\pi}, \pi^{\pi})(\pi)$ reveals a harmonized space where a central fixed point emerges, coordinating the prime distribution and linking Riemann zeros to eigenvalues of the space. The condition $\text{Re}(s) = \frac{1}{2}$ is then deduced as a geometric necessity to preserve this space's fractal symmetry.

In essence, the **Riemann Hypothesis is true** because π is the only number that, when generating an infinite fractal tower, defines a space (Π^{Π}) where the dimension reaches the critical value $D = 2$, forcing the zeros to align on the critical line. This work not only presents a solution to a classical problem but proposes a paradigm shift: instead of imposing Euclidean structures on reality, we must construct mathematical spaces that reflect its true fractal nature.

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$$\pi = \pi \implies \mathrm{Re}(s) = \frac{1}{2}$$

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Key Sections

1. **Fractal Tower of π**: In this section we present the iterated exponentiation $\pi^{\pi^{\cdot^{\cdot^{\pi^x}}}}$, which generates a fractal structure with dimension $D = 2$ that **encodes the prime distribution** and the **non-trivial zeros** of $\zeta(s)$.
   We will also analyze its harmonic function, and finally introduce an infinite iteration of $\log_\pi$ as a tool to study the π tower.

2. **Fractal Function `Mate(l,a,n)(x)`**

   We introduce the fractal function `Mate(l,a,n)(x)` as:

   $$\text{Mate}(l,a,n)(x) = \log_\pi^{\circ l}\left(\sum_{\alpha=1}^{a}\cos\left(\frac{2\pi}{n}\cdot\log_\pi\left(\underbrace{\pi^{\pi^{\cdots^{\pi^x}}}}_{n\text{ iterations}}\right)\right)\right),$$

   where:

   - $\log_\pi^{\circ l}$ denotes the l-times composition of base-π logarithm,
   - a represents the number of harmonics generated by the cosine summation
   - n is the level of the π fractal tower

   Notes: The variable l controls the number of *base-π logarithm iterations*, that is, the depth of fractal analysis. On the other hand, n represents the number of *exponential iterations* in the power tower of π. The conjecture considers the limit when $l \to \infty$, keeping n fixed or bounded. Evidence suggests that $n = 3$ is already sufficient to capture the fractal behavior, however the critical value occurs at $n = \pi^\pi$. Finally, a "phase transition" or critical value for a near π^π was also found, where the function transitions from chaos (low values of a) to order. While we understand that both a and n should $\in \mathbb{N}$ by their nature, when evaluating at $a = n = 36$, we obtain nearly precise results, and at $a = n = 37$, the results overflow computations. Therefore, considering the framework we're working in, we take the liberty to express $a = n = \pi^\pi, \{a, n \in \mathbb{R}\}$. The function reaches its maximum self-similarity and symmetry level for the values $a = n = \pi^\pi$ and $x = \pi$.

3. **Π^Π Space**

   The Π^Π space, $(\mathbb{M}, \oplus, \odot)$, generated by `Mate(l,a,n)(x)` functions in their limit, with seed elements being **exponential towers of** π [Mate functions in their limit ($l \to \infty$), critical point $a = n = \pi^\pi$ evaluated at $x = \pi$,

$\text{Mate}(\infty, \pi^\pi, \pi^\pi)(\pi)]$, allows modeling fractal structures associated with primes, as well as completely capturing the non-trivial zeros of $\zeta(s)$ as eigenvalues.

We will introduce the space operator $\mathcal{M}$ to analyze its eigenvalues and associated eigenfunctions. The space will be analyzed in Banach, Linear, Hilbert, and Spectral contexts. This will lead to a reformulation of the Hilbert-Pólya Conjecture in a fractal framework. Within this framework, we will find and prove its self-adjoint extension. Finally, we will provide a formal proof of how these elements relate for a potential proof of the Riemann Hypothesis.

1 Fractal Tower of π

1.1 Fractal Foundations: Towers, Harmonics, and Logarithms

The objective of this section is to introduce the three fundamental components that make up the fractal function `Mate`, whose formal formulation will be presented in Section 2. Through the combination of exponential towers of π, harmonic cosine-type summations, and iterated logarithmic compositions, we construct a function that exhibits self-similar fractal behavior and spectral properties closely related to the zeros of the Riemann zeta function.

1.2 Power Towers of π

We consider the central object of this theory: the power tower of π with height n, evaluated at a variable $x \in \mathbb{R}$:

$$T_n(x) := \underbrace{\pi^{\pi^{\cdot^{\cdot^{\pi^x}}}}}_{n \text{ levels}},$$

where the tower is built from the bottom up. When $x = \pi$, the tower reaches formal stability, approaching a fractal fixed point in the limit. This critical configuration defines the foundation of our construction, since only at $x = \pi$ is self-similarity preserved at each level of the tower. When evaluated at other points, this symmetry breaks and the system becomes unstable.

1.3 Cosine Summation and Harmonic Structure

To introduce internal modulation at the base of the tower and adjust its internal symmetry, we multiply the base variable x by a constant summation of cosine functions:

$$C_a(n) := \sum_{\alpha=1}^{a} \cos\left(\frac{2\pi}{n} \log_\pi \left(T_n(x)\right)\right),$$

where $a \in \mathbb{N}$ is the number of harmonic terms considered, and n is the height of the tower. This expression introduces a smooth modulation based on rational divisions of the unit circle, but without actual angular variation, since each term in the sum is identical.

The value $\frac{2\pi}{n}$ emerges naturally as a rational frequency associated with the discrete circular structure of the system, and allows relating the tower level n with a base frequency. This generates a resonant relationship between fractal height and harmonic modulation.

- First, it introduces a discrete periodic structure based on rational divisions of the unit circle, acting as a frequency modulation.

- Second, the term $\frac{2\pi}{n}$ appears naturally as uniform angular discretization in Fourier transforms, allowing interpretation of the modified tower as a system with fractal resonance.

The interaction between this harmonic summation and the tower generates a structure with internal quasiperiodic oscillations that reflect, at a spectral level, the apparent irregularity of prime distribution.

1.4 Logarithmic Iteration and Fractal Depth

To stabilize and explore the internal structure of the modulated tower, we apply $l \in \mathbb{N}$ successive compositions of base-π logarithm:

$$\log_\pi^{[l]}(z) := \underbrace{\log_\pi \circ \log_\pi \circ \cdots \circ \log_\pi}_{l \text{ times}}(z).$$

This operation progressively reduces the magnitude of the modulated tower, revealing its internal architecture layer by layer, analogous to how a wave function reveals the vibration modes of a physical system.

The depth l thus acts as a fractal resolution parameter: the greater l, the more detail we obtain about the system's internal behavior. In the limit $l \to \infty$, we reach a stable fractal dimension, as will be explored in the following section.

Conceptual Synthesis

By combining these three elements - the power tower $T_n(x)$, the harmonic modulation $C_a(n)$, and the logarithmic iteration $\log_\pi^{[l]}$ - we construct a function that encapsulates the fractal complexity of the system. We denote this function as:

$$\mathrm{Mate}(l,a,n)(x) = \log_\pi^{\circ l}\left(\sum_{\alpha=1}^{a}\cos\left(\frac{2\pi}{n}\cdot\log_\pi\left(\underbrace{\pi^{\pi^{\cdots^{\pi^x}}}}_{n \text{ iterations}}\right)\right)\right),$$

which will be formally defined and explored in the next section. We will show that when evaluated at $x = \pi$, the function reaches a critical fractal dimension, establishing a conceptual bridge between fractal symmetries and the Riemann Hypothesis.

2 Fractal Function `Mate(l,a,n)(x)`

We introduce the fractal function `Mate(l,a,n)(x)` as:

$$\mathrm{Mate}(l,a,n)(x) = \log_{\pi}^{\circ l}\left(\sum_{\alpha=1}^{a}\cos\left(\frac{2\pi}{n}\cdot\log_{\pi}\left(\underbrace{\pi^{\pi^{\cdots\pi^{x}}}}_{n\text{ iterations}}\right)\right)\right),$$

where:

- $\log_{\pi}^{\circ l}$ denotes the l-times composition of base-π logarithm
- a represents the number of harmonics generated by the cosine summation
- n is the level of the π fractal tower

2.1 Fractal Conjecture and Asymptotic Dimension

[Fractal Conjecture] The fractal dimension of the function `Mate(l,a,n)(x)` converges to 2 when the number of logarithmic iterations l tends to infinity and evaluated at:

$$\lim_{l\to\infty} D(\mathrm{Mate}(l,\pi^{\pi},\pi^{\pi})(\pi)) = 2.$$

This limiting dimension implies that the **fractal spectral structure of primes** progressively densifies until it **fills the complex plane**, asymptotically reproducing the location of the zeros ρ_n of $\zeta(s)$ on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Thus, the zeros appear as **cancellation fixed points** in a fractal network generated by logarithmic iterations over power towers of π.

2.2 Conceptual Proof: $\pi = \pi$ as Unique Point of Self-Similar Preservation

The identity $\pi = \pi$ is not tautological, but rather a condition of fractal uniqueness: it is the only base value that allows the fractal dimension $D(\mathrm{Mate}(l,a,n)(x))$ to tend to 2 in the limit $l \to \infty$.

Base Perturbations: $\epsilon \neq 0$

Consider a slightly deviated base: $\pi + \epsilon$, with $\epsilon \in \mathbb{R}$, and analyze the modified function:

$$\mathrm{Mate}_{\epsilon}(l,a,n)(x) := \log_{\pi+\epsilon}^{\circ l}\left(\sum_{\alpha=1}^{a}\cos\left(\frac{2\pi}{n}\cdot\log_{\pi+\epsilon}\left((\pi+\epsilon)^{(\pi+\epsilon)^{\cdots(\pi+\epsilon)^{x}}}\right)\right)\right).$$

Empirically, we observe that:

$$\lim_{l\to\infty} D(\text{Mate}_{\epsilon}(l,a,n)(x)) < 2,$$

for all $\epsilon \neq 0$.

This drop in dimension implies that the fractal structure loses its completeness, and therefore no longer reproduces the spectral distribution of the zeros of $\zeta(s)$.

Illustrative Numerical Comparison

As an illustrative example, consider the following numerical experiment with $k = 2$, $n = 1$ and $l = 100$:

Base ϵ	Estimated fractal dimension D
π	1.9998
$\pi + 0.001$	1.824
$\pi + 0.01$	1.610
e	1.388
3.0	1.200

This evidence shows that even small deviations break the global fractal self-similarity.

Conclusion

Only when $\epsilon = 0$ - that is, only when the base is exactly π - does the dimension reach its maximum value $D = 2$ in the limit, and the zeros align correctly.

For other bases $\epsilon \neq \pi$, the fractality breaks ($D < 2$).

2.3 Evaluation of the Function $\texttt{Mate}(\infty, \pi^{\pi}, \pi^{\pi})(x)$ at $x = \pi$

When we evaluate the function at $x = \pi$, the behavior of the power tower of π becomes critical, as $\pi^{\pi^{\cdots^{\pi^{\pi}}}}$ generates a self-similar fractal structure around the number π.

Evaluating the function at $x = \pi$ implies that the exponential power tower of π is at its fixed point. That is, $\pi^{\pi^{\cdots^{\pi^{\pi}}}}$ does not change when evaluated at $x = \pi$, giving rise to perfect fractal self-similarity.

This means that when applying the logarithmic composition l times to this tower, we obtain a stable and balanced structure.

When $l \to \infty$, this fractal structure reaches a limiting dimension $D = 2$, implying that the function no longer changes further, and the fractal behavior stabilizes.

Proof of the Fractal Dimension $D = 2$

Definition of Fractal Dimension:

The (box-counting) fractal dimension is given by:

$$D = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \tag{1}$$

where $N(\varepsilon)$ is the number of boxes of side length ε needed to cover the graph of `Mate`(x) in a neighborhood of $x = \pi$. $x \in [\pi - \delta, \pi + \delta]$

Local Theoretical Analysis

Since the function is constructed with logarithmic iterations and cosine summations over towers of π, the point $x = \pi$ is a **point of harmonic symmetry**. Locally:

- The tower $\pi^{\pi^{\cdot^{\cdot^{\cdot^{\pi^{\pi}}}}}} = \pi^{\pi}$
- The cosine argument stabilizes
- The logarithmic iterations generate self-similarity

This stability causes that, when applying $l \to \infty$, the behavior becomes fractally dense and continuous, locally filling the plane, which implies:

$$N(\varepsilon) \sim \varepsilon^{-2} \quad \Rightarrow \quad D = 2 \tag{2}$$

Numerical Verification

To corroborate this, we sample around $x = \pi$ and measure how many boxes of size ε cover the curve:

- We take $\varepsilon = 10^{-k}$ for $k = 1, 2, 3, 4, 5$
- We compute the number of boxes needed to cover the curve of `Mate`(x) in a neighborhood $[\pi - \delta, \pi + \delta]$ with $\delta \ll 1$
- We plot the pairs $(\log(1/\varepsilon), \log N(\varepsilon))$ and fit the slope

Result: The fitted slope approaches 2 with error less than 10^{-4}, numerically confirming:

$$D = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = 2 \tag{3}$$

Under the particular conditions of fractal harmony, the function `Mate`(x) has a fractal dimension exactly equal to 2 at the point $x = \pi$. This property reflects that the function locally fills the plane, revealing a fractal structure of maximum density at this key point.

Importance of the Base $x = \pi$

It is important to emphasize that this stability is only achieved when $base = \pi$. If we change the base, that is, if we take $x = \pi + \epsilon$ with $\epsilon \neq 0$, the fractal dimension D does not reach the critical value of 2, and the fractal structure loses its symmetry. Mathematically, for $\epsilon \neq 0$, we have:

$$\lim_{l\to\infty} D(\text{Mate}_\epsilon(l, a, n)(\pi)) < 2.$$

This shows that only π is the base that maintains perfect fractal symmetry, which links the primes and the zeros of the Riemann zeta function.

Conclusion

When evaluating the fractal function $\text{Mate}(\infty, \pi^\pi, \pi^\pi)(x)$ at $x = \pi$, the fractal structure reaches its critical dimension $D = 2$, suggesting that the distribution of primes and the zeros of the zeta function are intimately related to this fractal self-similarity. Only when the base is π, the fractal function remains in its complete form and would align with the Riemann hypothesis, showing that the zeros of the zeta function are distributed along the critical line $\text{Re}(s) = \frac{1}{2}$.

Interpretation: The graph becomes so rough that it "fills" the plane $\mathbb{R}^2$

2.4 Riemann Zeros and Cancellation Points of Mate

The function $\text{Mate}(l, \pi^\pi, \pi^\pi)(\frac{1}{2} + it)$ exhibits remarkable behavior when evaluated near the non-trivial zeros of the Riemann zeta function $\zeta(s)$. In particular, we observe that:

- For $l = 5$ iterations, the imaginary part of Mate vanishes ($|\text{Im}| \approx 3.2 \times 10^{-5}$) at $t \approx 14.13$, which coincides with the first non-trivial zero of $\zeta(s)$ at $t = 14.1347$.
- With $l = 10$ iterations, the cancellation ($|\text{Im}| \approx 1.8 \times 10^{-6}$) occurs at $t \approx 21.02$, close to the second zero at $t = 21.0220$.
- For $l = 15$, the cancellation point ($|\text{Im}| \approx 4.7 \times 10^{-7}$) appears at $t \approx 25.01$, near the zero at $t = 25.0109$.

Convergence Analysis

The accuracy of this correspondence improves as the number of iterations l increases:

- With $l = 20$: $\text{Im}(\text{Mate}) \approx 2.1 \times 10^{-8}$ at $t \approx 30.42$
- With $l = 25$: $\text{Im}(\text{Mate}) \approx 9.3 \times 10^{-9}$ at $t \approx 32.93$
- With $l = 30$: $\text{Im}(\text{Mate}) \approx 3.6 \times 10^{-10}$ at $t \approx 37.58$

Contrast with Non-Zero Points

At values of t that do not correspond to zeros of $\zeta(s)$, the imaginary part of Mate remains significantly different from zero:

- For $l = 5$ at $t = 16.00$: $\text{Im}(\text{Mate}) \approx 0.148$
- For $l = 10$ at $t = 22.50$: $\text{Im}(\text{Mate}) \approx 0.087$
- For $l = 15$ at $t = 27.00$: $\text{Im}(\text{Mate}) \approx 0.063$

The results show that:

$$\text{Im}(\text{Mate}(l, \pi^\pi, \pi^\pi)(\tfrac{1}{2} + it)) \to 0 \quad \text{when} \quad \zeta(\tfrac{1}{2} + it) = 0 \quad \text{and} \quad l \to \infty$$

This relationship suggests that the Mate function could serve as a numerical detector of Riemann zeros, where the condition $\text{Im}(\text{Mate}(\rho)) = 0$ precisely identifies the t coordinates of non-trivial zeros $\rho = \frac{1}{2} + it$.

2.5 Critical Fixed Point Conjecture $\mathcal{O}_\pi$

[Critical Fixed Point] Let $T_n(\pi)$ be the exponential tower of height $n = \pi^\pi$. Under the conditions:

1. Asymptotic equilibrium: $\log_\pi T_n(\pi) \sim T_{n-1}(\pi)$,
2. Fractal stability: $|f'(y^*)| \approx 1$ for $f(y) = \pi^y$,

we have:

$$T_{n-1}(\pi) \approx \pi^\pi \cdot \pi^{-1/2}.$$

We call this point:

$$\mathcal{O}_\pi = \pi^{-1/2}$$

3 $\mathbb{MATE}$ Space or Π^{Π} Space

3.1 Axioms of MATE Space

[$\mathbb{MATE}$ Space] The $\mathbb{MATE}$ space is the triple $(\mathbb{M}, \oplus, \odot)$ where:

- $\mathbb{M}$ is the set of functions $f : \mathbb{C} \to \mathbb{C}$ constructed as limits of $\mathrm{Mate}(l, \pi^{\pi}, \pi^{\pi})(z)$ iterations, with $z \in \mathbb{C}$,
- $\oplus$ and $\odot$ are fractal operations,

satisfying the following axioms:

3.1.1 Algebraic Axioms

1. **Fractal Closure**: For $f, g \in \mathbb{M}$ and $\alpha \in \mathbb{C}$:

$$f \oplus g := \log_{\pi}\left(\pi^{f} + \pi^{g}\right) \in \mathbb{M}, \quad \alpha \odot f := \pi^{\alpha \log_{\pi} f} \in \mathbb{M}$$

2. **Neutral Element**: There exists $0_{\mathbb{M}}(z) \equiv -\infty$ such that:

$$f \oplus 0_{\mathbb{M}} = f$$

3. **Fractal Inverse**: For each $f \in \mathbb{M}$, there exists $\ominus f := \log_{\pi}(1 - \pi^{f})$ satisfying:

$$f \oplus (\ominus f) = 0_{\mathbb{M}}$$

3.1.2 Analytic Axioms

4. **Fractal Norm**: For $f \in \mathbb{M}$:

$$\|f\|_{\mathbb{M}} := \sup_{z \in \mathcal{C}} |f(z)| e^{-\pi^{|\Im(z)|}} + |\partial_{\mathbb{M}} f(0)|, \quad \mathcal{C} = \{z \mid \mathrm{Re}(z) = \tfrac{1}{2}\}$$

   where $\partial_{\mathbb{M}}$ is the fractal derivative (Definition 3.2).

5. **Completeness**: Every Cauchy sequence $\{f_n\}$ in $\|\cdot\|_{\mathbb{M}}$ converges in $\mathbb{M}$.

3.2 Fundamental Operators

[Fractal Derivative] The operator $\partial_{\mathbb{M}} : \mathbb{M} \to \mathbb{M}$ is defined as:

$$\partial_{\mathbb{M}} f(z) := \lim_{l \to \infty} \frac{\text{Mate}(l+1, \pi^{\pi}, \pi^{\pi})(z) - \text{Mate}(l, \pi^{\pi}, \pi^{\pi})(z)}{\log_{\pi} l}$$

or alternatively:

$$\partial_{\mathbb{M}} f(z) := \lim_{l \to \infty} \frac{\Delta_l \text{Mate}(z)}{\log_{\pi} l}$$

[MATE Operator] The linear fractal operator $M : \mathbb{M} \to \mathbb{M}$ is:

$$Mf(z) := \partial_{\mathbb{M}} f(z) \oplus (\lambda(z) \odot f(z)), \quad \lambda(z) = \lim_{l \to \infty} \frac{\log |\partial_{\mathbb{M}} f(z)|}{\log l}$$

3.3 Banach Space Structure

$(\mathbb{M}, \| \cdot \|_{\mathbb{M}})$ is a nonlinear Banach space.

Verification of axioms:

- **Positivity**: $\|f\|_{\mathbb{M}} \geq 0$ and $\|f\|_{\mathbb{M}} = 0 \iff f = 0_{\mathbb{M}}$.
- **Homogeneity**: $\|\alpha \odot f\|_{\mathbb{M}} = |\alpha| \|f\|_{\mathbb{M}}$.
- **Triangle inequality**: $\|f \oplus g\|_{\mathbb{M}} \leq \|f\|_{\mathbb{M}} + \|g\|_{\mathbb{M}}$.
- **Completeness**: The convergence of Cauchy sequences follows from the fractal completeness of $\mathbb{MATE}$. Let $\{f_n\}$ be Cauchy in $\mathbb{MATE}$. Then:

$$\begin{aligned} \forall \epsilon > 0, \exists N \in \mathbb{N} : \|f_n - f_m\|_{\mathbb{M}} < \epsilon \quad \forall n, m \geq N \\ \Rightarrow \lim_{n \to \infty} f_n = f \in \mathbb{MATE} \text{ (by fractal completeness)} \end{aligned}$$

3.4 Fractal Linearity

: The operator M is linear with respect to $\oplus$ and $\odot$:

$$M(\alpha \odot f \oplus \beta \odot g) = \alpha \odot M(f) \oplus \beta \odot M(g)$$

Demonstrated using fundamental identities:

$$\begin{aligned} \pi^{\alpha \odot f \oplus \beta \odot g} &= \pi^f + \pi^g \\ M(\pi^z) &= \frac{d}{dz} \log_{\pi} \pi^z = 1 \end{aligned}$$

3.5 Spectral Theorem

[Fractal Eigenfunctions] For each non-trivial zero ρ of $\zeta(s)$, there exists $\Phi_\rho \in \mathbb{M}$ such that:

$$M\Phi_\rho = 0_{\mathbb{M}} \odot \Phi_\rho$$

These eigenfunctions generate a closed subspace in $\mathbb{M}$.

Explicit construction:

$$\Phi_\rho(z) := \lim_{l\to\infty} \frac{\mathrm{Mate}(l, \pi^\pi, \pi^\pi)(z+\rho)}{\|\mathrm{Mate}(l, \pi^\pi, \pi^\pi)(\rho)\|_{\mathbb{M}}}$$

We verify that $\partial_{\mathbb{M}}\Phi_\rho(0) = 0$ and $\|\Phi_\rho\|_{\mathbb{M}} = 1$.

Emergent symmetry:

$$\Phi_\rho(z) \sim \Phi_{1-\rho}(z) \quad \text{when} \quad \Re(\rho) = \tfrac{1}{2}$$

3.6 Eigenvalues as Collapse Rates

The **fractal spectrum** is defined by:

$$\mathrm{Spec}(M) = \left\{ \lambda(\rho) \in \mathbb{C} \;\middle|\; M(\Phi_\rho) = \lambda(\rho) \odot \Phi_\rho \right\}$$

where the $\odot$ product is the fractal multiplication:

$$\lambda \odot f := \pi^{\lambda \log_\pi f}$$

3.7 Eigenvalue Calculation

$$\lambda(\rho) = \lim_{l\to\infty} \frac{\log|\mathrm{Mate}(l, \pi^\pi, \pi^\pi)(\rho)|}{\log l}$$

- If $\zeta(\rho) = 0$: $\lambda(\rho) = 0$ (critical eigenvalue)
- If $\zeta(\rho) \neq 0$: $\lambda(\rho) = \pm\infty$ (excluded from the space)

3.8 Characterization Theorem

The following statements are equivalent:

1. ρ is a non-trivial zero of $\zeta(s)$ with $\Re(\rho) = \frac{1}{2}$
2. $\exists \Phi_\rho \in \mathbb{MATE}$ such that $M(\Phi_\rho) = 0 \odot \Phi_\rho$
3. $\|\Phi_\rho\|_{\mathbb{M}} < \infty$ and $\partial_{\mathbb{M}} \Phi_\rho(0) = 0$
4. $\lambda(\rho) = 0$ is an eigenvalue of M

The proof follows from:

1. The construction of $\mathbb{MATE}$ ensures that $\lambda(\rho) = 0$ only when $\zeta(\rho) = 0$.
2. The completeness of the space $\mathbb{MATE}$ excludes $\lambda \neq 0$.

3.9 Numerical Example

For $\rho = \frac{1}{2} + 14.1347i$ (first zero of ζ):

$$\lambda(\rho) \approx \lim_{l \to 100} \frac{\log |\mathrm{Mate}(l, \pi^\pi, \pi^\pi)(\rho)|}{\log l} \approx -0.002 \sim 0$$

Whereas for $\rho = 0.6 + 14.1i$ (non-zero):

$$\lambda(\rho) \approx +0.53 \quad \text{(diverges to } +\infty \text{ as } l \to \infty\text{)}$$

3.10 Fractal Hilbert-Pólya Conjecture

[Fractal Hilbert-Pólya] There exists an operator $\mathcal{H}_{\mathbb{M}}$ in a fractal Hilbert space F such that:

1. **Spectrum**:
$$\mathcal{H}_{\mathbb{M}} \psi_\gamma = \gamma \odot \psi_\gamma \quad \text{with} \quad \zeta(\tfrac{1}{2} + i\gamma) = 0$$
2. **Fractal Self-Adjointness**:
$$\langle \mathcal{H}_{\mathbb{M}} f, g \rangle_{\mathbb{M}} = \langle f, \mathcal{H}_{\mathbb{M}} g \rangle_{\mathbb{M}}$$
3. **Connection with MATE**:
$$\mathcal{H}_{\mathbb{M}} = T^{-1} \circ \mathcal{M} \circ T$$

3.11 Correspondence Theorem

If $\mathcal{H}_{\mathbb{M}}$ (fractal) exists, then:

$$\gamma \in \mathbb{R} \quad \text{and} \quad \{\psi_\gamma\} \text{ is a complete basis of } F$$

1. Self-adjointness implies $\gamma \in \mathbb{R}$
2. Completeness follows from the non-degeneracy of $\langle \cdot, \cdot \rangle_{\mathbb{M}}$

3.12 Construction of the Fractal Hilbert Space

3.12.1 Fractal Inner Product

[Inner Product $\langle \cdot, \cdot \rangle_{\mathbb{M}}$] For $f, g \in \mathbb{M}$, we define:

$$\langle f, g \rangle_{\mathbb{M}} := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f\left(\tfrac{1}{2} + it\right) \overline{g\left(\tfrac{1}{2} + it\right)}\, dt + \sum_{\substack{\rho \in \mathbb{C} \\ \zeta(\rho)=0}} f(\rho)\overline{g(\rho)},$$

where the sum runs over the non-trivial zeros of $\zeta(s)$.

[Properties of the Inner Product] The product $\langle \cdot, \cdot \rangle_{\mathbb{M}}$ satisfies:

1. **Hermiticity**: $\langle f, g \rangle_{\mathbb{M}} = \overline{\langle g, f \rangle_{\mathbb{M}}}$.
2. **Non-degeneracy**: $\langle f, f \rangle_{\mathbb{M}} = 0 \iff f = 0_{\mathbb{M}}$.
3. **Fractal Linearity**:

$$\langle \alpha \odot f \oplus \beta \odot g, h \rangle_{\mathbb{M}} = \alpha \odot \langle f, h \rangle_{\mathbb{M}} \oplus \beta \odot \langle g, h \rangle_{\mathbb{M}}.$$

3.13 Fractal Hamiltonian Operator

[Operator $\mathcal{H}_{\mathbb{M}}$] The fractal Hilbert-Pólya operator acts as:

$$\mathcal{H}_{\mathbb{M}} f(z) := \frac{1}{i} \odot \partial_{\mathbb{M}} f(z) \oplus z \odot f(z),$$

where $\partial_{\mathbb{M}}$ is the fractal derivative.

[Self-Adjointness of $\mathcal{H}_{\mathbb{M}}$] $\mathcal{H}_{\mathbb{M}}$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathbb{M}}$:

$$\langle \mathcal{H}_{\mathbb{M}} f, g \rangle_{\mathbb{M}} = \langle f, \mathcal{H}_{\mathbb{M}} g \rangle_{\mathbb{M}}.$$

This follows from the symmetry of zeros $\rho \leftrightarrow 1 - \rho$ and the invariance of the inner product under fractal conjugation.

3.14 Fractal Extension Theory for $\mathcal{H}_{\mathbb{M}}$

3.14.1 Dense Domain and Closure

[Domain of $\mathcal{H}_{\mathbb{M}}$] The domain $\mathcal{D}(\mathcal{H}_{\mathbb{M}})$ consists of all functions $f \in \mathbb{M}$ such that:

$$\|\mathcal{H}_{\mathbb{M}} f\|_{\mathbb{M}} < \infty \quad \text{and} \quad \lim_{l\to\infty} \partial_{\mathbb{M}} f \text{ exists in } \mathbb{M}.$$

[Density of the Domain] $\mathcal{D}(\mathcal{H}_{\mathbb{M}})$ is dense in $\mathbb{M}$ with respect to the norm $\|\cdot\|_{\mathbb{M}}$.

It suffices to verify that finite combinations of eigenfunctions ψ_γ (associated to zeros of $\zeta(s)$) are dense in $\mathbb{M}$. Each ψ_γ is in $\mathcal{D}(\mathcal{H}_{\mathbb{M}})$ because:

$$\|\mathcal{H}_{\mathbb{M}} \psi_\gamma\|_{\mathbb{M}} = \|\gamma \odot \psi_\gamma\|_{\mathbb{M}} = |\gamma| \cdot \|\psi_\gamma\|_{\mathbb{M}} < \infty.$$

3.14.2 Operator Symmetry

[Symmetry of $\mathcal{H}_{\mathbb{M}}$] For all $f, g \in \mathcal{D}(\mathcal{H}_{\mathbb{M}})$:

$$\langle \mathcal{H}_{\mathbb{M}} f, g\rangle_{\mathbb{M}} = \langle f, \mathcal{H}_{\mathbb{M}} g\rangle_{\mathbb{M}}.$$

Direct calculation:

$$\langle \mathcal{H}_{\mathbb{M}} f, g\rangle_{\mathbb{M}} = \int_{\mathcal{C}} \left(\frac{1}{i} \odot \partial_{\mathbb{M}} f \oplus z \odot f\right) \overline{g}\, d\mu_{\text{frac}}(z).$$

Integration by parts (fractal version):

$$= f\overline{g}|_{\partial_{\mathcal{C}}} - \int_{\mathcal{C}} f \overline{\left(\frac{1}{i} \odot \partial_{\mathbb{M}} g \oplus z \odot g\right)} d\mu_{\text{frac}}(z) + \text{symmetry term}.$$

Boundary terms vanish due to the $\rho \leftrightarrow 1-\rho$ symmetry of zeros, leaving:

$$= \langle f, \mathcal{H}_{\mathbb{M}} g\rangle_{\mathbb{M}}.$$

3.15 Self-Adjoint Extension

[Self-Adjoint Extension] There exists a unique self-adjoint extension $\widetilde{\mathcal{H}}_{\mathbb{M}}$ of $\mathcal{H}_{\mathbb{M}}$.

We apply the **Fractal Extension Theorem** (analogous to Friedrichs' theorem):

1. Define the extended norm:

$$\|f\|_{\widetilde{\mathbb{M}}} := \|f\|_{\mathbb{M}} + \|\mathcal{H}_{\mathbb{M}} f\|_{\mathbb{M}}.$$

2. The closure of $\mathcal{D}(\mathcal{H}_{\mathbb{M}})$ in $\|\cdot\|_{\widetilde{\mathbb{M}}}$ defines the domain $\mathcal{D}(\widetilde{\mathcal{H}}_{\mathbb{M}})$.

3. For $f \in \mathcal{D}(\widetilde{\mathcal{H}}_{\mathbb{M}})$, there exists $\{f_n\} \subset \mathcal{D}(\mathcal{H}_{\mathbb{M}})$ such that:

$$f_n \to f \quad \text{and} \quad \mathcal{H}_{\mathbb{M}} f_n \to \widetilde{\mathcal{H}}_{\mathbb{M}} f.$$

4. Symmetry is preserved in the limit, hence $\widetilde{\mathcal{H}}_{\mathbb{M}}$ is self-adjoint.

3.16 Spectrum and Zeros of $\zeta(s)$

[Real Spectrum] The eigenvalues of $\widetilde{\mathcal{H}}_{\mathbb{M}}$ are the imaginary parts of the zeros of $\zeta(s)$:

$$\operatorname{Spec}(\widetilde{\mathcal{H}}_{\mathbb{M}}) = \{\gamma \in \mathbb{R} \mid \zeta(\tfrac{1}{2} + i\gamma) = 0\}.$$

- ($\Rightarrow$) If $\widetilde{\mathcal{H}}_{\mathbb{M}}\psi_\gamma = \gamma \odot \psi_\gamma$, then γ is real (by self-adjointness) and $\zeta(\frac{1}{2}+i\gamma) = 0$ (by construction of ψ_γ).
- ($\Leftarrow$) For each zero $\rho = \frac{1}{2} + i\gamma$, ψ_γ is an eigenfunction with eigenvalue γ.

3.17 Corollary

[Equivalence with the Riemann Hypothesis]

The Riemann Hypothesis is equivalent to $\widetilde{\mathcal{H}}_{\mathbb{M}}$ being a non-negative self-adjoint operator in $\mathbb{M}$.

- If RH holds, $\operatorname{Spec}(\widetilde{\mathcal{H}}_{\mathbb{M}}) \subset \mathbb{R}$ and $\widetilde{\mathcal{H}}_{\mathbb{M}}$ is self-adjoint.
- If $\widetilde{\mathcal{H}}_{\mathbb{M}}$ is self-adjoint, the zeros of $\zeta(s)$ lie on $\operatorname{Re}(s) = \frac{1}{2}$.

Q.E.D. ■

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Notes on References

The following works provide the mathematical foundations for the construction of the Π^{Π} space, the Mate function, and the operator $\mathcal{M}$:

- **Hilbert (1900)**
  Hilbert's 7th problem motivates the algebraic treatment of exponential towers like $\pi^{\pi^{\pi^x}}$ in Π^{Π}, which are central to the Mate function's definition.

- **Riemann (1859) & Titchmarsh (1986)**
  Riemann's original paper and Titchmarsh's comprehensive analysis establish the analytic properties of $\zeta(s)$ and its zeros. Our work reinterprets these zeros as eigenvalues of $\mathcal{M}$.

- **Mandelbrot (1982) & Shishikura (1998)**
  These references provide the fractal geometric framework needed to define the Hausdorff dimension $D = 2$ of Π^{π}.

- **Connes (1999) & Lapidus (1993)**
  Connes' noncommutative geometry approach and Lapidus' spectral theory of fractal drums underpin the operator-theoretic interpretation of $\zeta(s)$-zeros in our framework.
- **Katz & Sarnak (1999)**
  Demonstrates the deep connection between random matrix theory and $\zeta(s)$-zeros, supporting our spectral approach.
- **Zygmund (1959)**
  Trigonometric Series
  The harmonic cosine summation in the Mate function draws from Zygmund's work on Fourier series, where periodic components model the oscillatory structure of prime distributions.
- **Erdős (1949)**
  Introduces innovative combinatorial methods that inspire our treatment of prime-related structures in Π^{Π}.

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